Small Quasi-Kernels in Hairy Tournaments

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Abstract

Let $D = (V, E)$ be a digraph. A vertex set $K \subseteq V$ is a *quasi-kernel* of D if K is an independent set in D and for every vertex $v \in V \setminus K$, v is at most a distance of 2 from K. It is a well-known result of Chvátal and Lovász that every digraph has a quasi-kernel. In 1976, P. L. Erdős and L. A. Székely conjectured that if every vertex of D has a positive in-degree, then D has a quasi-kernel of size at most $|V|/2$. A *tournament* is obtained from a complete graph by assigning a direction to each edge, and a *hairy tournament* is a digraph whose deletion of all sink vertices yields a tournament, where a *sink vertex* is a vertex of zero out-degree. A *core* is the largest tournament of a hairy tournament. In this work, we study the size of a quasi-kernel in a hairy tournament and support the Erdős-Székely conjecture for hairy tournaments such that each vertex of its core is joined to at most two sink vertices.

Theorem

Let G be an n-vertex hairy tournament and $T \subseteq G$ be the core of G. If $\left|N_G^+(v) \cap (V(G) \setminus V(T)\right| \leq 2$ for every $v \in V(T)$, then G has a quasi-kernel of size at most $\frac{n}{2}$.
This Theorem implies that given an n-vertex hairy tournament in which the core is connected to no more than two sink vertices, one can find an independent set K of G satisfying two properties: for any $a \in V(G) \setminus K$ it holds that $dist_G(K, a) \leq 2$, and the number of vertices in K will be at most $\frac{n}{2}$.

Methods

In essence, our proof is an application of induction on the number of edges of G, or equivalently, by taking a counterexample G with the smallest number of edges to the statement, and then finding a way to reduce the graph to one G^* with a smaller number of edges. As G^* is no longer a counterexample to the statement, a desired quasi-kernel K^* of G^* can be found. We then modify K^* to get a desired quasi-kernel for G. In order to accomplish this, we begin by investigating the structures of a counterexample with the smallest number of edges to the statement. We prove two essential properties regarding the in-degree values of sink vertices and how those sink vertices relate to the tournament part of the graph.

Keywords: Quasi-Kernel, Erdős-Székely Conjecture, Hairy Tournament